Mathematics-II – X20001

(2nd Sem Civil PDDC 2013)

Assignment-5, Beta and Gamma Functions)

1. Prove that $\Gamma(m+1) = m\Gamma(m)$. Also prove $\Gamma(m+1) = m!$ when m is positive integer.

2. Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

3. Prove that $\displaystyle\int_0^\infty e^{-x^2}dx = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$.

4. Evaluate following integrals using Beta and Gamma functions

   (a) $\displaystyle\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy.$

   (b) $\displaystyle\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy.$

   (c) $\displaystyle\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx.$

   (d) $\displaystyle\int_0^1 x^5\left[\log\frac{1}{x}\right]^3 dx.$

   (e) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^5}}dx.$

   (f) $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^4}}dx.$

   (g) $\displaystyle\int_0^1 x^3(1-\sqrt{x})^5 dx.$

   (h) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}}dx.$

5. Prove that

   (a) $\dfrac{\beta(m+1,n)}{m} = \dfrac{\beta(m,n+1)}{n} = \dfrac{\beta(m,n)}{m+n}.$

   (b) $\beta(m+1,n) + \beta(m,n+1) = \beta(m,n).$

   (c) $\displaystyle\int_b^a (x-a)^m(b-x)^n dx = (b-a)^{m+n+1}\beta(m+1,n+1).$

6. Compute $\beta(1.5, 2.5)$, $\beta\left(\dfrac{9}{2}, \dfrac{7}{2}\right)$.

7. Show that $\displaystyle\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}d\theta = \pi.$

ASSIGNMENT : 5 Beta & Gamma

Q : 1 Prove that $\Gamma m+1 = m\Gamma m$. Also prove $\Gamma m+1 = m!$ when m is positive integer.

Soln :- Using defn of $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$

So, $\Gamma m+1 = \int_0^\infty e^{-x} x^m dx$

Integration by parts

$$= \left[x^m \int e^{-x} dx \right]_0^\infty - \int_0^\infty \left[\frac{d}{dx}(x^m) \int e^{-x} dx \right] dx$$

$$= \left[x^m \left(\frac{e^{-x}}{-1} \right) \right]_0^\infty - \int_0^\infty \left[m x^{m-1} \left(\frac{e^{-x}}{-1} \right) \right] dx$$

$$= [0 - 0] + m \int_0^\infty e^{-x} . x^{m-1} dx$$

$$= m \Gamma m$$

2) $\Gamma m+1 = m!$

$$\Rightarrow \Gamma m+1 = m \Gamma m$$

$$= m \Gamma (m-1)+1$$

$$= m . (m-1) \Gamma m-1$$

$$= m(m-1) \Gamma (m-2)+1$$

$$= m(m-1) . (m-2) \Gamma m-2 \ldots\ldots$$

$$= m(m-1)(m-2) \ldots\ldots 1 \Gamma 1$$

Now $\Gamma 1 = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} dx$

$$= \left[\frac{e^{-x}}{-1} \right]_0^\infty = \left[\frac{e^{-\infty} - e^0}{-1} \right] = 1$$

So, $\Gamma m+1 = m(m-1)(m-2) \ldots\ldots 1$

$$= 1 \; 2 . 3 \ldots\ldots (m-2)(m-1) m$$

$$= m!$$

Q12. Prove that $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Soln :- Proof, we know that $\Gamma n = \int_0^\infty e^{-t} \, t^{n-1} \, dt$

Taking $t = x^2$

$\therefore dt = 2x \, dx$

$\therefore x = 0 \Rightarrow t = 0$ & $x \to \infty \Rightarrow t \to \infty$

$$\therefore \Gamma n = \int_0^\infty e^{-x^2} (x^2)^{n-1} \, 2x \, dx$$

$$= 2 \int_0^\infty e^{-x^2} \, x^{2n-2} \, x \, dx$$

$$= 2 \int_0^\infty e^{-x^2} \, x^{2n-1} \, dx \quad \text{—— (1)}$$

lly $\Gamma m = 2 \int_0^\infty e^{-y^2} \, y^{2m-1} \, dy$

Now $\Gamma n \, \Gamma m = \left[2 \int_0^\infty e^{-x^2} x^{2n-1} \, dx\right] \cdot \left[2 \int_0^\infty e^{-y^2} y^{2m-1} \, dy\right]$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, x^{2n-1} \, y^{2m-1} \, dx \, dy$$

Now changing in to polar cordinates by taking

$x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$ & r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

$$\therefore \Gamma n \, \Gamma m = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} \, r \, dr \, d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} r^{2n-1} \cos^{2n-1}\theta \, r^{2m-1} \sin^{2m-1} r \, dr \, d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} r^{2n-1+2m-1+1} \cos^{2n-1}\theta \, \sin^{2m-1}\theta \, dr \, d\theta$$

$$= 4\left[\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr\, d\theta\right]$$

$$= \left[2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta\right]\left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right]$$

$\therefore \Gamma n \, \Gamma m = B(m,n) \times \Gamma m+n$ ($\because$ By proof of Beta f^n in ①)

$$\therefore B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Q:3 Prove that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi} = \Gamma 1/2$

By def^n of Gamma f^n

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\therefore \Gamma 1/2 = \int_0^{\infty} e^{-x} x^{1/2-1} dx = \int_0^{\infty} e^{-x} x^{-1/2} dx$$

Taking $x = y^2 \Rightarrow dx = 2y\,dy$ when $x=0 \Rightarrow y=0$ & $x \to \infty \Rightarrow y \to \infty$

$$\therefore \Gamma 1/2 = \int_0^{\infty} e^{-y^2} (y^2)^{-1/2} 2y\,dy$$

$$= 2\int_0^{\infty} e^{-y^2} y^{-1} y\, dy$$

$$= 2\int_0^{\infty} e^{-y^2} dy$$

S.lly. $\Gamma 1/2 = 2\int_0^{\infty} e^{-x^2} dx$

$$(\Gamma 1/2)(\Gamma 1/2) = \left(2\int_0^{\infty} e^{-y^2} dy\right)\left(2\int_0^{\infty} e^{-x^2} dx\right)$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-x^2} e^{-y^2} dx\, dy$$

Now changing into polar co-ordinates by taking

$x = r\cos\theta$ & $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ & $dxdy = r\,dr\,d\theta$

where

θ varies from 0 to $\pi/2$ &

r varies from 0 to ∞

$$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\tfrac{1}{2}\,dt\,d\theta$$

$\left(\because \text{Let } r^2 = t \Rightarrow 2r\,dr = dt \Rightarrow r\,dr = \tfrac{1}{2}dt, \text{ where } r=0 \Rightarrow t=0 \text{ \& } r\to\infty \Rightarrow t=\infty\right)$

$$= 2\int_0^{\pi/2}\int_0^{\infty} e^{-t}\,dt\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} d\theta$$

$$= 2\int_0^{\pi/2}\left[0-\frac{1}{(-1)}\right]d\theta = 2\int_0^{\pi/2} 1\,d\theta$$

$$= 2\left[\theta\right]_0^{\pi/2}$$

$$= 2\left[\pi/2 - 0\right] = \pi$$

$$\therefore \Gamma\tfrac{1}{2} = \sqrt{\pi} = \int_0^{\infty} e^{-x^2}dx.$$

Q:4 Evaluate following integrals using Beta and Gamma functions:

a) $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

Taking $\log_e 1/y = t \Rightarrow e^t = 1/y$ & $y = e^{-t} \Rightarrow dy = e^{-t}(-1)dt$

when $y=1 \Rightarrow t \to \infty$ & $y=0 \Rightarrow t \to \infty$

$$\therefore \quad = \int_{\infty}^{0} (t)^{n-1} \left(-e^{-t}\right) dt$$

$$= \int_{0}^{\infty} e^{-t} \, t^{n-1} \, dt$$

$$= \Gamma n$$

b) $$\int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}} \, dy.$$

soln: Let $I = \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}} \, dy$

Taking $\frac{1}{1+y} = t \quad \therefore \left(\frac{1}{1+y}\right)^{p+q} = t^{p+q} = \frac{1}{(1+y)^{p+q}}$.

$\therefore 1+y = 1/t \quad \therefore y = 1/t - 1 \quad$ i.e $y = \frac{1-t}{t}$

Also, $dy = -1/t^2 \, dt \quad y=0 \Rightarrow t=1, \; y \to \infty \Rightarrow t=0$

$$\therefore I = \int_1^0 \left(\frac{1-t}{t}\right)^{q-1} t^{p+q} \left(-\frac{1}{t^2} dt\right)$$

$$= \int_0^1 \frac{(1-t)^{q-1}}{t^{q-1}} \, t^{p+q} \, \frac{1}{t^2} \, dt$$

$$= \int_0^1 t^{p+q-q+1-2} (1-t)^{q-1} \, dt$$

$$= \int_0^1 t^{p-1} (1-t)^{q-1} \, dt$$

$$= \beta(m,n) = \beta(p,q)$$

c) $$\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} \, dx.$$

$$I = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} \, dx$$

$$= \int_0^1 \left[\frac{x^{p-1}}{(1+x)^{p+q}} + \frac{x^{q-1}}{(1+x)^{p+q}} \right] dx$$

$$= \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}} \, dx + \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} \, dx$$

Taking $x = 1/z$ in 2nd term. $dx = -1/z = dz$

$\therefore x = 0 \Rightarrow z \to \infty$ & $x = 1 \Rightarrow z = 1$

$$\therefore \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} \, dx = \int_\infty^1 \frac{(1/z)^{q-1}}{(1+1/z)^{p+q}} (-1/z^2 \, dz)$$

$$= \int_1^\infty \frac{1/z^{q-1}}{\frac{(z+1)^{p+q}}{z^{p+q}}} \cdot 1/z^2 \, dz = \int_1^\infty \frac{z^{p+q-q+1-2}}{(1+z)^{p+q}} \, dz$$

$$= \int_1^\infty \frac{z^{p-1}}{(1+z)^{p+q}} \, dz = \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \, dx. \quad \text{—(1)}$$

$$\therefore \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}} + \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \, dx$$

$$= \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \, dx$$

$= \beta(m,n)$ ($\because$ from Example b)

d) $\int_0^1 x^5 \left[\log \frac{1}{x}\right]^3 dx.$ —(1)

Put $\log_e 1/x = t$

So, $e^t = 1/x \to x = e^{-t}$ $\therefore dx = -e^{-t} \, dt$

when $x=0 \Rightarrow e^{-t}=0=e^{-\infty}$ so, $t=\infty$

$x=1 \Rightarrow e^{-t}=1=e^{0}$ so, $t=0$

so, ①

$$\int_{\infty}^{0} (e^{-t})^5 \cdot (t)^3 (-e^{-t}\, dt)$$

$$= \int_0^{\infty} e^{-5t} \cdot t^3 \, e^{-t} \, dt$$

$$= \int_0^{\infty} e^{-t(5+1)} \cdot t^3 \, dt$$

$$= \int_0^{\infty} t^3 \, e^{-t-5t} \, dt$$

$$= \int_0^{\infty} t^3 \cdot e^{-6t} \, dt$$

Now taking $6t = K$ $\therefore t = K/6$ $\therefore dt = 1/6\, K\, dK$

and $t \Rightarrow 0 \Rightarrow K \Rightarrow 0$ & $t=\infty \Rightarrow K=\infty$

$$= \frac{1}{6^3}\int_0^{\infty} e^{-K} K^{4-1} dK = \frac{1}{6^3} \cdot \Gamma 4 = \frac{1}{36}$$

e) $\int_0^1 \frac{x}{\sqrt{1-x^5}} dx$

Put $x^5 = t$ $\therefore 5x^4 dx = dt$ $\therefore dx = \frac{dt}{5t^{4/5}}$

when $x=0$, $t=0$, when $x=1$, $t=1$

$$\therefore \int_0^1 \frac{t^{1/5}}{\sqrt{1-t}} \cdot \frac{dt}{5t^{4/5}}$$

$$= \frac{1}{5} \int_0^1 \frac{t^{1/5-4/5}}{(1-t)^{1/2}} dt$$

$$= \frac{1}{5} \int_0^1 t^{-3/5} (1-t)^{-1/2} dt$$

$$= \frac{1}{5} \int_0^1 t^{2/5 - 1} (1-t)^{1/2 - 1} \, dt$$

$$= \frac{1}{5} \beta\left(\frac{2}{5}, \frac{1}{2}\right)$$

$$= \frac{1}{5} \frac{\Gamma 2/5 \, \Gamma 1/2}{\Gamma 2/5 + 1/2} = \frac{\frac{1}{5} \Gamma 2/5 \, \Gamma 1/2}{\Gamma 9/10} = \frac{\frac{1}{5} \sqrt{\pi} \, \Gamma 2/5}{\Gamma 9/10}$$

f) $\int_0^1 \frac{1}{\sqrt{1+x^4}} \, dx$

Let $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^4 = t$ $\therefore x = t^{1/4}$ $\therefore dx = 1/4 \, t^{-3/4} \, dt$

$x = 0 \Rightarrow t = 0$ & $x = 1 \Rightarrow t = 1$

$$\therefore I = \int_0^1 \frac{1}{(1-t)^{1/2}} \, 1/4 \, t^{-3/4} \, dt$$

$$= 1/4 \int_0^1 t^{-3/4} (1-t)^{-1/2} \, dt$$

$$= 1/4 \int_0^1 t^{1/4 - 1} (1-t)^{1/2 - 1} \, dt$$

$$= 1/4 \, \beta(1/4, 1/2)$$

$$= 1/4 \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 1/4 + 1/2} = \frac{1/4 \, \Gamma 1/4 \sqrt{\pi}}{\Gamma 3/4}$$

g) $\int_0^1 x^3 (1 - \sqrt{x})^5 \, dx$

Let $I = \int_0^1 x^3 (1 - \sqrt{x})^5 \, dx$

Let $\sqrt{x} = t$

$\therefore x = t^2$

$\therefore dx = 2t \cdot dt$

when $x = 0 \Rightarrow t = 0$ & $x = 1 \Rightarrow t = 1$

$$\therefore \int_0^1 x^3 (1-\sqrt{x})^5 dx = \int_0^1 (t^2)^3 (1-t)^5 dt \, 2(t)$$

$$= 2\int_0^1 t^7 (1-t)^5 dt$$

$$= 2\int_0^1 t^{8-1} (1-t)^{6-1} dt = 2B(8,6)$$

$$= \frac{2\,\Gamma 8\,\Gamma 6}{\Gamma 8+6} = \frac{2\,7!\,6!}{13!}$$

$$= \frac{\cancel{2}\,\cancel{7!} \cdot \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}{13 \times \cancel{12} \times 11 \times \cancel{10} \times \cancel{9} \times \cancel{8}} = \frac{1}{858}$$

2 3 4

h) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} dx$

Sol^n: Let $1-x^n = t$ $\therefore x = (1-t)^{1/n}$ $\therefore dx = -1/n\,(1-t)^{\frac{1-n}{n}} dt$

$t = 1$ when $x = 0$ & $t = 0$ when $x = 1$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^n}} = -\frac{1}{n}\int_1^0 \frac{(1-t)^{\frac{1-n}{n}}}{t^{1/2}} dt$$

$$= \frac{1}{n}\int_0^1 t^{1/2-1} (1-t)^{\frac{1}{n}-1}$$

$$= \frac{1}{n} B\left(\frac{1}{2}, 1/n\right)$$

$$= \frac{1}{n} \frac{\Gamma 1/2\, \Gamma 1/n}{\Gamma 1/n + 1/2} = \frac{\sqrt{\pi}}{n} \frac{\Gamma 1/n}{\Gamma 1/n + 1/2}$$

Q:5 Prove that

a) $$\frac{B(m+1,n)}{m} = \frac{B(m,n+1)}{n} = \frac{B(m,n)}{m+n}$$

Soln :- $$\frac{B(m+1,n)}{m} = \frac{\Gamma(m+1)\,\Gamma(n)}{m\,(\Gamma(m+1+n))}$$

$$= \frac{\cancel{m}\,\Gamma(m)\,\Gamma(n)}{\cancel{m}\,(m+n)\,\Gamma(m+n)} = \frac{\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{1}{m+n}\,\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{B(m,n)}{m+n} \quad \text{——— (1)}$$

$$\frac{B(m,n+1)}{n} = \frac{\Gamma(m)\,\Gamma(n+1)}{n\,\Gamma(m+n+1)} = \frac{\Gamma(m)\,\cancel{n}\,\Gamma(n)}{\cancel{n}\,(m+n)\,\Gamma(m+n)}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{B(m,n)}{m+n} \quad \text{——— (2)}$$

from (1) & (2)

$$\frac{B(m+1,n)}{m} = \frac{B(m,n+1)}{n} = \frac{B(m,n)}{m+n}$$

b) $B(m+1,n) + B(m,n+1) = B(m,n)$

soln :- LHS $= B(m+1,n) + B(m,n+1)$

$$= \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\,\Gamma(m)\,\Gamma(n)}{\Gamma(m+1+n)} + \frac{\Gamma(m)\,n\,\Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma m \, \Gamma n \, [m+n]}{\Gamma(m+n)+1} = \frac{\Gamma m \, \Gamma n \, [\cancel{m+n}]}{(\cancel{m+n}) \, \Gamma m+n} = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \beta(m, n)$$

d) $\int_b^a (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \beta(m+1, n+1)$

Soln:- $\int_a^b (x-a)^m (b-x)^n \, dx$

Put $x-a = z$ & $x = z+a \Rightarrow dx = dz$

when $x = a \Rightarrow z = 0$

$x = b \Rightarrow z = b-a$

Also $b-x = b-z-a$ Again put $z = (b-a) \cdot y$

$\Rightarrow dz = (b-a) \, dy$

when $z = 0 \Rightarrow y = 0$ & $z = b-a \Rightarrow y = 1$

$$= \int_0^1 [(b-a)y]^m [(b-a)-(b-a)y] (b-a) \, dy$$

$$= \int_0^1 (b-a)^m . y^m (b-a)^n . [1-y]^n . (b-a) \, dy$$

$$= (b-a)^{m+n+1} . \beta(m+1, n+1) = \text{RHS.}$$

Q:6 Compute $\beta(1.5, 2.5)$, $\beta(\frac{9}{2}, \frac{7}{2})$

i) $\beta(1.5, 2.5) = \frac{\Gamma 3/2 \, \Gamma 5/2}{\Gamma 3/2 + 5/2} = \frac{\Gamma 3/2 \, \Gamma 5/2}{\Gamma 4} = \frac{1/2 \, \Gamma 1/2 \;\; 3/2 \, 1/2 \, \Gamma 1/2}{3!}$

$$= \frac{1 \cdot \sqrt{\pi} \cdot \cancel{3} \cdot 1 \sqrt{\pi}}{(\cancel{3} \times 2 \times 1)(2 \times 2 \times 2)}$$

$$= \frac{\pi}{16}$$

ii) $B\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{\Gamma 9/2 \, \Gamma 7/2}{\Gamma 9/2 + 7/2} = \frac{\Gamma 9/2 \, \Gamma 7/2}{\Gamma 8}$

$$= \frac{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma 1/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma 1/2}{7!}$$

$$= \frac{\cancel{7} \cdot \cancel{5} \cdot \cancel{3} \cdot 1 \cdot \sqrt{\pi} \; 5 \cdot 3 \cdot 1 \cdot \sqrt{\pi}}{(\cancel{7} \times 6 \times \cancel{5} \times 4 \times \cancel{3} \times 2 \times 1)(2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)}$$

$$= \frac{\cancel{15}^{5} \pi}{\cancel{6}_{2} \times 4 \times 2 \times 128} = \frac{5\pi}{2048}$$

Q. 7 Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}\, d\theta = \pi$

Soln:- Let $I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}\, d\theta.$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta \, d\theta \times \int_0^{\pi/2} \cos^{-1/2}\theta \cdot \sin^0\theta \, d\theta.$$

$$= \frac{1}{2} B\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) \times \frac{1}{2} B\left(\frac{0+1}{2}, \frac{-1/2+1}{2}\right)$$

$$= \frac{1}{4} B\left(\frac{3}{4} \; \frac{1}{2}\right) B\left(1/2 \; 1/4\right)$$

$$= 1/4 \, \frac{\Gamma 3/4 \, \Gamma 1/2}{\Gamma 3/4 + 1/2} \times \frac{\Gamma 1/2 \, \Gamma 1/4}{\Gamma 1/2 + 1/4} = 1/4 \, \frac{\Gamma 3/4 \, \sqrt{\pi}}{\Gamma 5/4} \cdot \frac{\Gamma 1/4 \, \sqrt{\pi}}{\Gamma 3/4}$$

$$= \frac{1}{4} \, \frac{\pi \, \Gamma 1/4}{1/4 \, \Gamma 1/4} = \pi = \text{R.S.H.}$$